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THE WEST COAST OF BRITAIN: STATISTICAL SELF-SIMILARITY VS. CHARACTERISTIC SCALES IN THE LANDSCAPE

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ABSTRACT

Mandelbrot (*Science*, 1967, **156**, 636–638) used the west coast of Britain as an example of a naturally occurring statistically self-similar fractal. Evidence from this study indicates that the west coast of Britain is not statistically self-similar over the range of scale of measurement, and that complexity reaches maxima at characteristic scales related to identifiable features on the coastline. A fractal analysis is conducted using the divider method, and although the resulting log–log plot of measured length against steplength appears linear, statistical tests for linearity strongly suggest that the coastline is not statistically self-similar. An angle measure technique (AMT) developed by the author to examine changes in line complexity with scale, shows that within the range of scale of measurement there are two peaks in complexity for the west coast of Britain, suggesting that two processes acting at different scales have influenced coastal development. The AMT is also used to identify differences in complexity between northern and southern sections of the coastline. Additionally, high r^2 values associated with regressions of log $L(G)$ against log G are shown to be insufficient evidence of statistical self-similarity, and apparently linear segments (fractal elements) often found in Richardson plots may contain systematic curvature revealed only by more rigorous tests for non-linearity.

KEY WORDS coastline; scale; fractal; complexity; angle measurement; statistical self-similarity

INTRODUCTION

Mandelbrot (1967, 1975, 1982) introduced the concepts of statistical self-similarity and fractional (fractal) dimension to the study of natural phenomena. A statistically self-similar line is one 'such that each ... portion can—in a statistical sense—be considered a reduced-scale image of the whole' (Mandelbrot, 1967). This implies that the complexity of a statistically self-similar line is independent of scale (i.e. constant over scale), and fractal dimension is a scale-independent measure of this complexity.

Fractal dimension of coastlines is usually measured with the divider method (for a review of fractal methods, see Burrough (1985)). This method involves measuring the length of a line repeatedly with divider steps of different lengths. The measured length ($L(G)$) of the line is equal to the product of the number of steps (N) and steplength (G). Fractal dimension is derived from the equation

$$L(G) = MG^{1-D} \tag{1}$$

where M is a constant and D is the fractal dimension. Fractal dimension can be calculated directly from the equation (Goodchild, 1980)

$$D = \log(N_2/N_1)/\log(G_1/G_2) \tag{2}$$

where N_1 and N_2 are the number of steps of lengths G_1 and G_2 needed to span a line. (More than two steplengths are normally employed, in which case D is usually estimated from a plot of log N against log G using least-squares linear regression analysis (D is equal to the absolute value of the slope of the relation).)

Using Richardson's (1961) data for the west coast of Great Britain, Mandelbrot used the coastline as an

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example of a naturally occurring statistically self-similar fractal, assigning it a fractal dimension of $D = 1{\cdot}25$ (Mandelbrot, 1967). Subsequently, the coastline has been considered by many to be the most popular example of a statistically self-similar fractal in nature, and it is often mentioned in the literature (Curl, 1986; Krantz, 1989; Turcotte, 1991).

In a re-examination of the west coast of Britain, Goodchild (1980) measured the length of the coastline with steplengths ranging from 10 to 1000 km and confirmed Mandelbrot's single value of $D = 1{\cdot}25$ for the coastline. However, in a similar analysis of the east coast of Britain, Goodchild identified two ranges of constant fractal dimension separated by a transition zone over which D varies. He recognized that the applicability of the concept of statistical self-similarity to natural phenomena is often limited to specific ranges of scale. These ranges of constant D were later called fractal elements (Orford and Whalley, 1983; Mark and Aronson, 1984). Shelberg *et al.* (1983), utilizing a computer-based divider walk algorithm, found that $D = 1{\cdot}267$ for the west coast of Britain. They attributed the difference in their estimate of D from previous estimates to the capturing of smaller-scale detail by digitization.

In calculating D with the divider method, the concept of statistical self-similarity (and fractal dimension) is based on linearity in the log–log relationship between the measured length $L(G)$ of a line and steplength G (or between log N and log G). More recently it has been demonstrated that most such apparently linear plots are in fact curvilinear (Andrle and Abrahams, 1989; Lam and Quattrochi, 1992; Andrle, 1994). In his original study, Richardson noted possible non-linearity in his log $L(G)$–log G plot for the west coast of Britain, stating that 'more evidence would be needed before one could say whether the deviations from the straight line are of any interest' (Richardson, 1961). Many researchers have turned to the fractal element model to account for such non-linearities (Orford and Whalley, 1983; Mark and Aronson, 1984; Klinkenberg and Goodchild, 1992). However, the fractal element model is also based on an assumption of statistical self-similarity (i.e. of linearity in log $L(G)$–log G plots) over limited ranges of scale. By using the fractal element model, one assumes that D is constant over identifiable ranges of scale (i.e. fractal elements), which are usually presumed to correspond to the domain of a particular process creating complexity in the landscape across that range of scale (Goodchild and Mark, 1987; Lam and Quattrochi, 1992). An alternative view is that processes create one or more maxima in complexity at particular (characteristic) scales, with complexity decreasing toward minima on either side. In this model, the relationship between complexity and scale is curvilinear, with one or more peaks and pits, each peak presumably corresponding to some complexity-producing process in the landscape.

In this study, the west coast of Britain is examined to determine whether it is statistically self-similar or if there exist characteristic scales (i.e. complexity varies with scale). The relationship between complexity and scale for the coastline is more closely studied using fractal analysis, and also with a non-fractal measure of complexity developed by the author specifically to examine changes in complexity with scale. The applicability of the concept of statistical self-similarity and the fractal element model to geomorphic lines is also examined.

ANALYSIS

The divider method

First, for comparison with Richardson's (1961) data, an analysis of the west coast of Britain was conducted using the six steplengths employed in his original study (Table I). For greater accuracy, the coastline was digitized at a high level of resolution producing a sequential file of 14 496 coordinates of the coastline (Figure 1). The two source maps are 1:625 000 scale Ordnance Survey Route Planning maps (south and north). The digitizer was set to incremental-radial mode in which coordinates are automatically collected when the cursor is moved a specified radial distance. This technique produces a relatively constant spacing of digitized coordinates which allows the use of a non-interpolating divider program. The mean spacing of the digitized points is 358 m, a value smaller than the level of cartographic generalization, estimated visually to be approximately 1000 m (the size of the smallest features commonly represented on the source maps, such as small bays and estuaries).

Results from this computer analysis of the coastline (Table I) are in close agreement with Richardson's

Table I. Values of steplength G and number of steps N for the west coast of Britain

Steplength, G (m)	Number of steps, N Richarson (1961)	Number of steps, N This study	Error in N (difference between estimates)
10 000	309·04	293·10	0·05
30 000	69·46	69·10	0·01
100 000	14·93	15·40	0·03
200 000	5·60	5·90	0·05
490 000	2·00	2·00	0·00
971 000	1·00	1·00	0·00

original figures. A Richardson plot (after Richardson, 1961) of log $L(G)$ against log G appears approximately linear (Figure 2), and $D = 1{\cdot}25$ for the new data, which is the same as Mandelbrot's value of D for the coastline. It can be inferred from this that differences in methodology between Richardson's manual use of the divider method and the computer-based implementation in this study do not significantly affect the results. Differences between estimates from this analysis and the value of $D = 1{\cdot}267$ derived by Shelberg *et al.* (1983) can most likely be attributed to the use of a different range or number of steplengths.

The six steplengths used in the previous analysis are insufficient to examine the log $L(G)$–log G relation for possible non-linearity. Therefore, a second fractal analysis was conducted, this time utilizing 166 steplengths logarithmically distributed from 2000 m to 100 000 m. A Richardson plot of the 166 points appears fairly

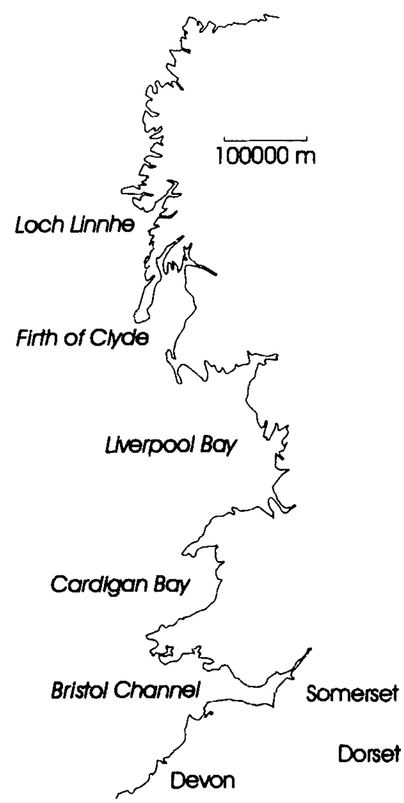

Figure 1. Outline map of the west coast of Britain

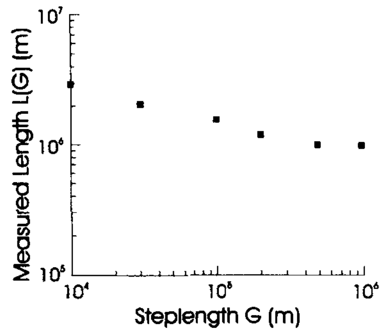

Figure 2. Plot of log $L(G)$ against log G for the west coast of Britain using six steplengths

linear (Figure 3), so according to usual practice, least-squares linear regression analysis was performed. The resulting equation

$$\log L(G) = 9{\cdot}36 - 0{\cdot}295 \log G \tag{3}$$

has an r^2 value of 0·99 ($t = -125{\cdot}25$). These results seem to indicate that the plot is linear (i.e. that the coastline is statistically self-similar), and that $D = 1{\cdot}295$. In order to test the linearity of the log $L(G)$–log G relation, a lack-of-fit test XLOF (Minitab, Inc., 1985) was applied to the data. The XLOF test checks for linearity by subdividing the data into three ranges (low, medium and high G values) and comparing the slopes of regression lines derived for these subsets. The lack-of-fit test revealed possible curvature in the plot ($P < 0{\cdot}01$) and a Durbin–Watson statistic of 0·16 ($P < 0{\cdot}01$) signifies positive serial correlation in the residuals. Both tests strongly suggest that the plot contains significant and systematic curvature. Examining the residuals from a regression is perhaps the best method for determining the linearity or non-linearity of a relationship (Mann, 1987). A plot of the standardized residuals of the above regression confirms the systematic nature of the curvature (Figure 3). This indicates that the coastline is not statistically self-similar over the range of scale examined.

It is possible that the plot contains one or more linear segments (fractal elements). The standard method for delineating fractal elements is first to identify breakpoints between fractal elements where D changes, and then to fit lines to each intervening straight segment (Mark and Aronson, 1984). To confirm that D is constant within each fractal element, each linear segment of the plot must then be tested for linearity as

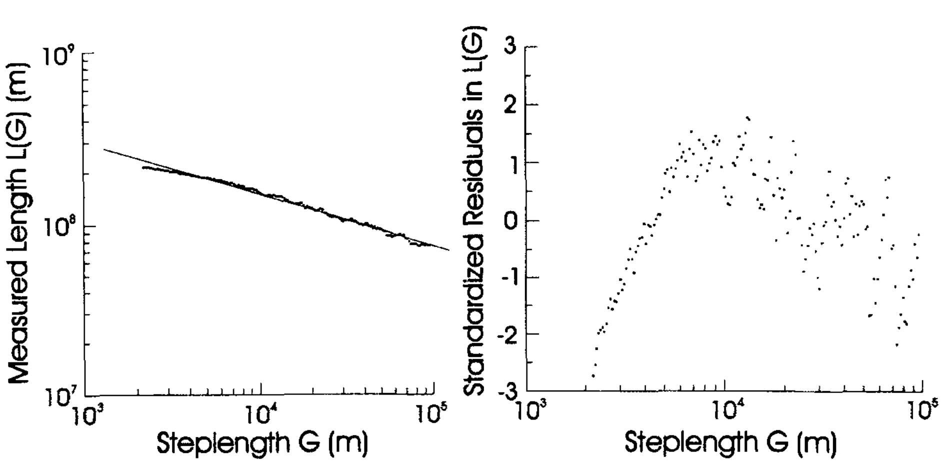

Figure 3. Plot of log $L(G)$ against log G for the west coast of Britain using 166 steplengths, and a plot of the standardized residuals from the regression of log $L(G)$ against log G

in the above analysis. However, as progressively smaller segments of the plot are chosen, at some point it is nearly inevitable that statistical tests for linearity will indicate that the section is linear. This is because the degree of scatter of points about the line will increase relative to the length of the line as the range of scale is reduced, eventually masking any non-linearity. This author is unaware of an established lower bound to the range of scale required to define a fractal element. Lacking this, the method cannot be considered a reliable indicator of statistical self-similarity of lines. One might argue that a close approximation of a non-linear relationship by a linear function is acceptable. However, in fractal analysis we are interested in the slope of the line, from which D is derived, rather than in simply predicting y from x as in most geomorphic studies involving linear approximation of a non-linear relation. The presence of even slight curvature in Richardson plots, while not significantly affecting the prediction of y from x can cause great variations in the slope of the line and, hence, in D (Andrle and Abrahams, 1989; Andrle, 1992).

Several additional deficiencies in fractal analysis exist. First, Richardson plots tend toward linearity because of spurious self-correlation that exists between $L(G)$ (which is equal to $N \times G$) and G (G is on both axes of the plots). Therefore, use of such plots to identify statistically self-similar features (by identifying linear segments) is unreliable because they tend to exaggerate the prevalence of statistical self-similarity in the landscape. Spurious self-correlation also results in r^2 values for regressions approaching unity, and is a violation of one of the assumptions of least-squares linear regression analysis (Mann, 1987), a method commonly used to estimate D. In this situation, high r^2 values are also a poor indicator of linearity of such plots, though they are often used as such in fractal analyses (Andrle, 1992).

Linear regression analysis is normally used for estimating y from x, a process for which the slope of the relation is estimated. In fractal analysis, estimating the slope is the primary goal. This is an important distinction with respect to the interpretation of the coefficient of determination. In point of fact, r^2 cannot tell you whether the unexplained variance is caused by random error (scatter) or by non-linearity in the relationship (Lewis-Beck, 1980). Indeed, the use of least-squares linear regression analysis requires the *a priori* assumption that the plot is linear. Linearity is a critical assumption of least-squares linear regression, which if violated biases parameter estimates (i.e. the slope coefficient) (Lewis-Beck, 1980). Therefore, even in the presence of high r^2 values, one must still test for linearity. Ironically, this author has found that significant curvature in Richardson plots can usually *only* be detected by statistical tests when r^2 is close to unity (i.e. when scatter, which hides systematic non-linearities, is minimal; Andrle, 1992), as are most Richardson plots (such as in Figure 3 for the west coast of Britain in this study).

Second, most methods employed to delineate the extent of individual fractal elements are highly subjective (Lam and Quattrochi, 1992). Varying the bounds of a fractal element will affect the determination of D for that range of scale. Third, complexity (and D) has been found to vary with scale for many natural features, or to be constant over only limited ranges of scale (Xu *et al.* 1993). In an extensive analysis of 55 DEM subsamples, Klinkenberg and Goodchild (1992) found many to lack statistical self-similarity. Fractal analysis is not applicable to these features, and it is difficult (if not impossible) to ascertain beforehand whether a particular example or over what ranges of scale a feature is statistically self-similar (Klinkenberg and Goodchild, 1992; Xu *et al.*, 1993). Fourth, no significance tests for statistical self-similarity exist in the literature (Xu *et al.*, 1993). Therefore, basing conclusions on results derived from untested assumption of statistical self-similarity is at best highly speculative. Last, what is usually of most interest to geomorphologists is the identification of trends and variations in the landscape. Fractal analysis, based on the concept of statistical self-similarity, is inherently not well suited to analysing such trends and variations in complexity with scale. Given the above, the use of an alternative methodology that is directed toward analysing variations in complexity with scale, and is not based on any assumption about the form of the relation between complexity and scale, is warranted.

A more promising fractal technique, multifractal analysis, is gaining favour. In this approach, fractal dimension can assume an infinite number of values, to allow for variations in complexity over scale (Tessier *et al.*, 1993). However, almost all applications of this technique involve the study of fluids or phenomena related to fluids such as turbulence, cloud patterns, rainfall, diffusion-limited aggregation, etc (e.g. Lovejoy and Schertzer, 1986; Stanley and Meakin, 1988; Tessier *et al.*, 1993). Until multifractals are applied more extensively, it is premature to state whether the technique will be useful in the study of landforms.

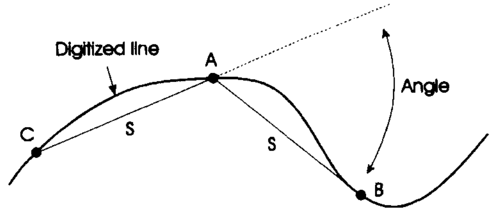

Figure 4. Illustration of the angle measure technique

The angle measure technique

The angle measure technique (AMT) was developed by this author (Andrle, 1994) expressly for measuring changes in the complexity of lines with scale. It provides an index of complexity calculated for individual scales instead of from multiple steplengths as with the divider method. Therefore, the angle measure avoids the *a priori* assumption in fractal analysis that the relative complexity of the line is constant (i.e. the line is statistically self-similar) over the range of scale of measurement for which a single value of D is calculated. The AMT is similar to cartographic line-generalization techniques involving measuring angles between points on a digitized line (for a review, see McMaster (1986)).

A FORTRAN program was written by the author to implement the AMT. First, a point is randomly chosen along a digitized line (point A, Figure 4). Two more points are then identified by measuring from this point forward and backward along the line. Points B and C are then placed as the digitized points closest to distance S from point A, and the supplementary angle of angle BAC is calculated. The procedure is repeated 500 times and the mean angle (MA) calculated for each (scale) S.

As the complexity of a line increases, so does the change in direction along a particular length of the line. Because MA is a measure of the change in direction over a length of line, it is a direct measure of line complexity. A plot of MA against log S is essentially a plot of the variation in line complexity with scale. From this plot, two measures describing local maxima in complexity can be derived. A moving-average algorithm is used to smooth the data in order to estimate the horizontal positions and heights of any peaks in the MA–log S plot. The horizontal position of a peak is the characteristic scale (S_c) of the line (i.e. a scale at which complexity as measured by MA reaches a local maximum value), and the peak's height defines the characteristic angle (A_c) attained at the characteristic scale. A_c is a measure of the *degree* of complexity attained by the line at that characteristic scale.

The AMT was applied to the west coast of Britain over a range of scale from 1000 m to 100 000 m, approximately the same range of scale (steplengths) used in the previous analysis. A plot of MA against log S (Figure 5A) shows that within the range of scale of measurement there are two clear peaks in complexity for the west

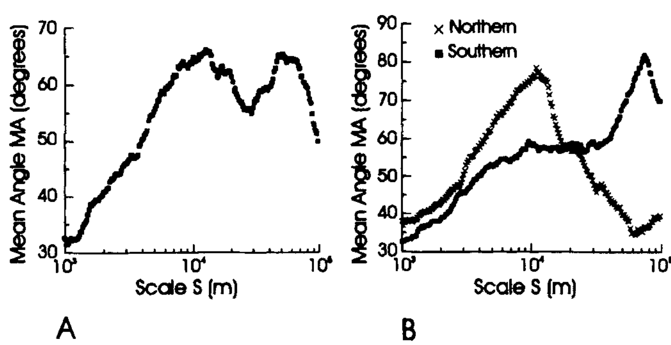

Figure 5. Plots of (A) mean angle against scale for the entire west coast of Britain, and (B) for the northern and southern sections of the coastline

coast of Britain, one at $S_c = 10\,800$ m and the other at $S_c = 56\,700$ m. Both have similar degrees of complexity at $A_c = 64°$ and $A_c = 63°$, respectively. The presence of two peaks in complexity for the coastline suggests that two distinct processes acting at different scales have influenced the development of the present form of the coastline.

Visual examination of the outline of the coastline (Figure 1) reveals the existence of large bays (i.e. the Bristol Channel, Liverpool Bay, etc.) that define the general outline of the coast. Numerous smaller bays and estuaries are superimposed upon these larger bays, and there is a dearth of features of intermediate scale. This suggests that the large and small bays correspond to the two peaks in complexity identified by the AMT (Figure 5A). The larger indentations in the coastline (e.g. the Bristol Channel, Cardigan Bay, Liverpool Bay, etc.) have been created primarily by coastal submergence and the presence of large-scale tectonic features including granite batholiths (Calder, 1972) and parallel grabens (Naylor, 1982). In contrast, the small indentations, on the order of 10 000 m in size, are submerged valleys originally cut by fluvial and glacial erosion (Goudie, 1990).

A closer visual examination of the west coast of Britain (Figure 1) reveals that the large indentations are mainly confined to the coasts of England and Wales. This fact allows further examination of variations in the complexity of the coastline. An MA–log S plot for the northern section of coast (north of Loch Linnhe) (Figure 5B) shows a single peak in complexity at $S_c = 10\,300$ m, which corresponds well with the characteristic scale of the left-hand peak in the plot for the entire coast ($S_c = 10\,800$ m). In contrast, the plot for the southern section of coast has a single peak in complexity at $S_c = 72\,200$ m (comparable to $S_c = 56\,700$ m for the entire coast). The low peak or shelf on the lefthand limb can be explained by the existence of (less numerous) small bays on the southern section of coast. These results lend credence to the suggestion made previously that the two peaks in Figure 5A correspond to the large and small bays. It is also apparent from this analysis that complexity varies not only with scale but with position along the coastline.

CONCLUSIONS

The evidence presented in this study indicates that the west coast of Britain is not statistically self-similar over the range of scale of measurement, and that complexity varies in a continuous and systematic manner with scale. Specifically, complexity reaches maxima at certain (characteristic) scales related to identifiable features on the coastline, and can be linked at least qualitatively to their formative processes.

Several issues regarding the use of fractal analysis in geomorphology are illuminated by this study. First, high r^2 values associated with regressions of log $L(G)$ against G (as in Figure 2), while often seen as indicators of linearity (and, hence, as evidence of statistical self-similarity), are misleading and more rigorous tests of linearity of Richardson plots should be undertaken in the future. Second, given that fractal dimension must be calculated for a range of scale (i.e. with at least two steplengths), it is by default assumed constant over that range of scale. Thus, fractal analysis is not well suited to analysing geomorphic phenomena whose complexity varies over scale, such as the west coast of Britain in this study. Third, a comparison of the Richardson plot in Figure 3 with the AMT plots (Figure 5A, B) indicates that the apparently linear segments (fractal elements) often found in Richardson plots might contain significant systematic curvature revealed only by more rigorous tests for non-linearity than have commonly been used in fractal analysis. This author suggests that any application of fractal analysis to geomorphic lines be accompanied by rigorous testing of the linearity of the Richardson plots from which D is derived.

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